导数极限定理是说：如果f(x)在x0的某领域内连续，在x0的去心邻域内可导，且导函数在x0处的极限存在(等于a)，则f(x)在x0处的导数也存在并且等于a。

证明：$f(x)$ 在 $U(x_0,\delta)$ 连续，在 $U^\circ(x_0,\delta)$ 可导。

$f'(x_0)$ 在 x_0 处极限存在，则 $f(x)$ 在 x_0 可导。

① 取 $x_1 = x_0+\Delta x$，$\Delta x \to 0$。 (Lagrange 中值定理)

$$\frac{f(x_1)-f(x_0)}{\Delta x} = f'(\xi) \quad x_0 < \xi < x_0+\Delta x$$

令 $\Delta x \to 0$，

则 $f'_+(x_0) = f'(x_0^+)$

同理 $f'_-(x_0) = f'(x_0^-)$

而 $f'(x_0)$ 在 x_0 处极限存在。

故 $f'_+(x_0) = f'(x_0^-) = f'(x_0) = k$，故可导。

$f \in D(1,+\infty)$ 且 $\lim\limits_{x\to+\infty}(f(x)+xf'(x)\ln x) = a$ 证：$\lim\limits_{x\to\infty} f(x) = a$

证：$f(x)+xf'(x)\ln x$ | 而 $f(x) = \dfrac{f(x)\ln x}{\ln x}$

$= \dfrac{\frac{1}{x}f(x)+f'(x)\ln x}{\frac{1}{x}}$ | $x\to\infty$

| $\dfrac{f(x)\ln x}{\ln x}$ 为 $\frac{*}{\infty}$ 型

$= \frac{(f(x)\ln x)'}{(\ln x)'}$ | 若 $\lim\limits_{x\to 0} f(x) = \frac{(f(x)\ln x)'}{(\ln x)'} = a$

$= a$

prove up

$f(x)\in C^2[-a,a]$, $f(0)=0$. 则 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{f'(0)}{2}$

使用带 Lagrange 余项的 Taylor 展开。在 $x=0$ 处

解：$\because f(x) = \overset{=0}{f(0)} + \frac{f'(0)}{1}\cdot x + \frac{f''(\xi)}{2}x^2$. 令 $x = \frac{k}{n^2}$,

则 $f\left(\frac{k}{n^2}\right) = f'(0)\frac{k}{n^2} + \frac{f''(\xi_k)}{2}\left(\frac{k}{n^2}\right)^2$. 再进行 $k=1$ 到 n 累加得

$\sum\limits_{k=1}^{n} f\left(\frac{k}{n^2}\right) = f'(0)\sum\limits_{k=1}^{n}\frac{k}{n^2} + \sum\limits_{k=1}^{n}\frac{f''(\xi_k)}{2}\left(\frac{k}{n^2}\right)^2$, 当 $n\to+\infty$ 时

$= \frac{1}{2}f'(0)$. 则只要后面的一项为 0 即可

$\frac{1+\cdots+n}{n^2} = \frac{\frac{n^2+n}{2}}{n^2} = \frac{1}{2}\cdot\left(1+\frac{1}{n}\right)\to\frac{1}{2}$

想办法令 $f''(\xi_k)$ 有界从而放缩.

$\because \xi_k\in(0,a)$, $f(x)\in C^2[-a,a]$ ∴所导连续

故 $|f''(\xi_k)|\le M$

故 $\sum\limits_{k=1}^{n}\left|\frac{f''(\xi_k)}{2}\right|\left(\frac{k}{n^2}\right)^2 \le \frac{M}{2}\sum\limits_{k=1}^{n}\frac{k^2}{n^4}$

$\le M\cdot\frac{P(n^3)}{n^4}$

$=0$

$\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

prove up

$f(x)\in D^2[0,1]$, $f(0)=f(1)$, $|f''(x)|\le M$.

则有: $|f'(x)|\le \frac{M}{2}$, 其中 $x\in[0,1]$ 任意x成立

证 $f(0)=f(1)$. 则其Taylor展开也是相等的, 在x_0处用带lagrange

$$f(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2$$

$$f(0)=f(x_0)+f'(x_0)(-x_0)+\frac{f''(\xi_2)}{2}(-x_0)^2$$

$$f(1)=f(x_0)+f'(x_0)(1-x_0)+\frac{f''(\xi_1)}{2}(1-x_0)^2$$

$$f(1)-f(0)=f'(x_0)+\frac{f''(\xi_1)}{2}(1-x_0)^2-\frac{f''(\xi_2)}{2}x_0^2=0$$

即 $$f'(x_0)=\frac{f''(\xi_2)}{2}x_0^2-\frac{f''(\xi_1)}{2}(1-x_0)^2\le\frac{M}{2}\left(x_0^2-(1-x_0)^2\right)$$

$$=\frac{M}{2}(2x_0-1)\le\frac{M}{2}$$

$x_0\le 1$

$2x_0-1\le 1$

prove up

对于这个题目, 有一个结论为 $|f(x)|\le A$ $f''(x)\le B$.

$$|f'(x)|\le 2A+\frac{B}{2}$$

五、 (本题 15 分) 设 $f(x) = x^n(1-x)^n$,

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

计算并化简 $\frac{d}{dx}\big(F'(x)\sin x - F(x)\cos x\big)$.

$$\big(F'(x)\sin x\big)' = F''(x)\sin x + \cos x\, F'(x)$$

$$\big(F(x)\cos x\big)' = F'(x)\cos x - \sin x\, F(x).$$

原式 $= F''(x)\sin x + \sin x\, F(x)$

$= \big(F(x) + F''(x)\big)\sin x = f(x)\sin x$

$$\boxed{f(x) + (-1)^n \underbrace{f^{(2n+2)}(x)}_{=0}.}$$

六、 (本题 20 分) 设 $a=\sqrt[3]{3}$, $x_1=a$, $x_{n+1}=a^{x_n}$ $(n=1,2,\ldots)$. 证明: 数列 $\{x_n\}_{n=1}^{\infty}$ 极限存在，但不是 3.

$\ln 3$

$x_1 = 3^{\frac{1}{3}} < 3$

$x_2 = (3^{\frac{1}{3}})^{3^{\frac{1}{3}}} < (3^{\frac{1}{3}})^3 \cdot 3$

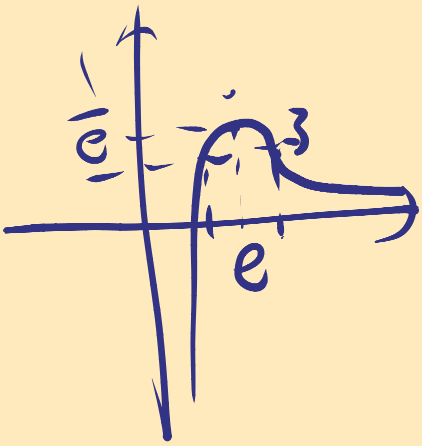
若 $x_{n-1} < 3$ 则 $x_n = (3^{\frac{1}{3}})^{x_{n-1}} < (3^{\frac{1}{3}})^3 < 3$

故 $\{x_n\}$ 有界为 3. 又 $\because 3^{\frac{1}{3}} > 1$ 故 $x_n \nearrow$.

$x_n \nearrow$ 有上界 x_n 极限存在

$x_{n+1} = a^{x_n}$ 设极限为 t

$t = a^t$ $\frac{\ln t}{t} = \frac{\ln 3}{3}$



$t = (3^{\frac{1}{3}})^t \to t \quad 3$

只要 $t < e$ 即可

$$x_1 = 3^{\frac{1}{3}} < 3^{\log_3 e} < e$$

$$x_2 = 3^{\frac{1}{3}x_1} < \underline{3^{\frac{e}{3}} < e} = 3^{\log_3 e}.$$

$$\because \frac{e}{3} < \log_3 e \Leftrightarrow \frac{e}{3} < \frac{\ln e}{\ln 3} \Leftrightarrow \frac{\ln e}{e} > \frac{\ln 3}{3}$$

故 $x_{n-1} < e \quad x_n \le 3^{\frac{e}{3}} < e$

$f(x) \in D^3(0,+\infty)$. $\lim\limits_{x\to\infty} f(x) = +\infty \quad \lim\limits_{x\to\infty} f^{(3)}(x) = 0$

即 $\lim\limits_{x\to\infty} f'(x) = \lim\limits_{x\to\infty} f'(x) = 0$

$$f(x_0+1) = f(x_0) + f'(x_0) + \frac{f''(x_0)}{2} + \frac{f'''(\xi_1)}{6}$$

$$f(x_0-1) = f(x_0) - f'(x_0) + \frac{f''(x_0)}{2} - \frac{f^{(3)}(\xi_2)}{6}$$

$x_0 \to +\infty \quad f(x_0+1) = f(x_0-1) = f(x_0) = \ldots$

x_0 对 [illegible]

$① + ② \Rightarrow f''(x) = 0 \quad ① - ② \quad f'(x) = 0$

$f(x) \in D^2(I)$ 对 $\forall [a,b] \subseteq I$ $\quad f'(a) = f'(b) = 0$

则 $\exists c \in [a,b]$ s.t. $\quad f''(c) \geq \frac{4}{(b-a)^2}|f(a) - f(b)|$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi_1)}{2}(x-a)^2 \quad ①$$

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(\xi_2)}{2}(x-b)^2 \quad ②$$

为使 $(x-a)^2 = (x-b)^2$. 令 $x = \frac{a+b}{2}$

则 ① - ② 得

$$f(a) - f(b) = \frac{(a-b)^2}{4}\left(\frac{f''(\xi_2)}{2} - \frac{f''(\xi_1)}{2}\right)$$

由泰勒公式得不等式

$$|f(a)-f(b)| \le \frac{(a-b)^2}{8}\cdot\frac{|f''(\xi_1)|+|f''(\xi_2)|}{2}$$

$$\le \frac{(a-b)^2}{8}\cdot f''(c).$$ c 为 ξ_1, ξ_2 中 $f''(\xi)$ 较大的那个.

故 $f''(c) \ge \frac{4}{(b-a)^2}|f(a)-f(b)|$

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young 不等式

若 $a, b \ge 0$. 且 $p, q > 0$ 且 $\frac{1}{p}+\frac{1}{q}=1$ (共轭指标) $ab \le \frac{a^p}{p}+\frac{b^q}{q}$

时. 利用 Jasen 不等式 $\frac{\ln a^p}{p}+\frac{\ln b^q}{q} \le \ln\left(\frac{a^p}{p}+\frac{b^q}{q}\right)$.

Holder 不等式

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$$

$$\frac{a_1b_1+\cdots+a_nb_n}{(a_1^p+\cdots+a_n^p)^{\frac{1}{p}}\cdot(b_1^q+\cdots+b_n^q)^{\frac{1}{q}}} \le \frac{1}{p}+\frac{1}{q}$$

$$\frac{\sum a_ib_i}{\sum\ \sum} \le \frac{\sum a_i^p}{p\sum a_i^p}+\frac{\sum b_i^q}{q\sum b_i^q}$$

令 $A_i = \frac{a_i}{\boxed{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}}}$ $B_i = \frac{b_i}{\boxed{\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}}}$ → 常数

$$\therefore A_iB_i \le \frac{a_i^p}{p\left(\sum_{i=1}^{n} a_i^p\right)}+\frac{b_i^q}{q\sum_{i=1}^{n} b_i^q}$$

事就后有 $\sum_{i=1}^{n} A_iB_i \le \frac{1}{p}+\frac{1}{q}=1$

即 $$\frac{\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i}{\sum_{i=1}^{n}(a_i^p)^{\frac{1}{p}} \cdot \sum_{i=1}^{n}(b_i^q)^{\frac{1}{q}}} \le 1$$ 得证.

Holder不等式的函数形式

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (g(x))^q dx\right)^{\frac{1}{q}}$$

令 $A=\frac{|f(x)|}{\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}}$ $B=\frac{|g(x)|}{\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}}$ 即可

利用young不等式后两边求积分即可

$$\int \frac{1}{1+x^3}dx \quad \int \frac{x}{1+x^3}dx$$

$$I+J=\int\frac{1+x}{1+x^3}dx$$

$$I-J=\int\frac{1-x}{1+x^3}dx=\int\frac{1-x+x^2-x^2}{1+x^3}dx$$

$$I=\int\frac{\cos x}{a\cos x+b\sin x}dx$$

$$J=\int\frac{\sin x}{a\cos x+b\sin x}dx$$

$$aI+bJ=\int 1dx$$

$$bI-aJ=\int\frac{1}{a\cos x+b\sin x}d(a\cos x+b\sin x)$$

配对

$\sin^n x$ $\cos^n x$ $\sec^n x$ $(\tan x)'$ $\tan^n x$ $(\tan^2 = \sec^2 - 1)$

$$\int \frac{1}{1+x^4}dx \ \& \ \int \frac{x^2}{1+x^4}dx$$

I J

$$I+J=\int \frac{1+x^2}{1+x^4}dx = \int \frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}dx$$

$$=\int \frac{1}{(x-\frac{1}{x})^2+2}\, d(x-\frac{1}{x})$$

$$\int \frac{1}{x(x^3+1)}dx = \int \frac{x^2}{x^3(x^3+1)}dx \cdots$$

$$\int \frac{P_n(x)}{(x-a)^n}dx = \int \frac{P_n(x)\ \text{Taylor展开}}{(x-a)^n}dx$$

$\int x^m(a+bx^n)^k dx$ 令 $t=x^n$, $\Rightarrow \frac{1}{n}\int t^q(a+bt)^k dt$

① $R(x, \sqrt[n]{\frac{ax+b}{cx+d}})$ 换元为t

配方

② $R(x, \sqrt{ax^2+bx+c})$

Euler替换

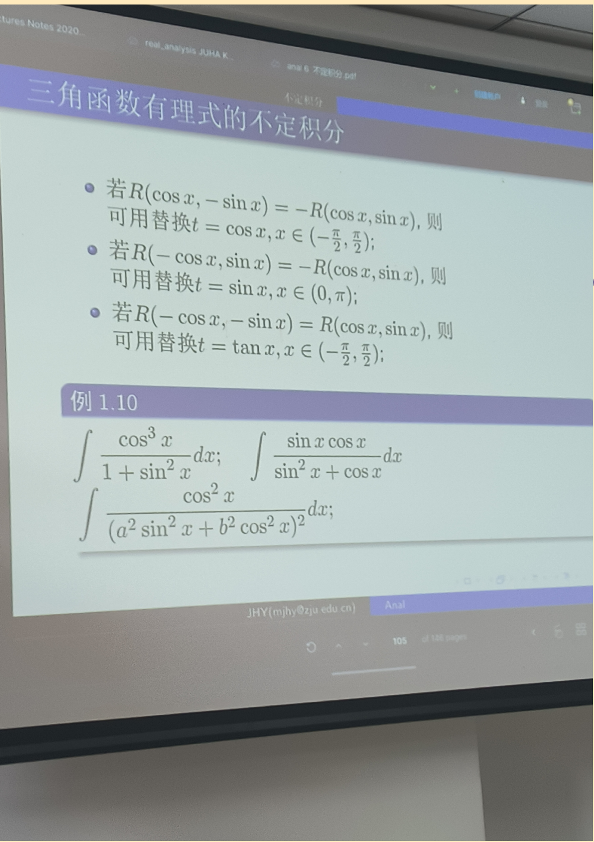
$a>0$, $\pm\sqrt{a}x+t$

$c>0$ $xt+\sqrt{c}$

有根 $(x-\alpha)t$

而对于 $\sqrt[n]{(ax+b)^i(cx+d)^k}$, $i+k=n$

$\Rightarrow (ax+b)\sqrt[n]{\frac{(cx+d)^k}{(ax+b)^k}}$ 同①



③ $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q(x)\sqrt{ax^2+bx+c} + \beta\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

求导伴系数

易错

$f(x)$ 在区间 I 可导

则 ① x_0 为 $f(x)$ 极大(小)值点，则 $\exists\,\delta>0$

$(x_0, x_0+\delta)\searrow(\nearrow)$ $\quad(x_0-\delta, x_0)\nearrow(\searrow)$

② $f'(x_0)>0$（<0）则 $\exists\,\delta$ $x\in U(x_0,\delta)$，$f'(x_0)\nearrow$

都是错的（考虑无限振荡的函数）.

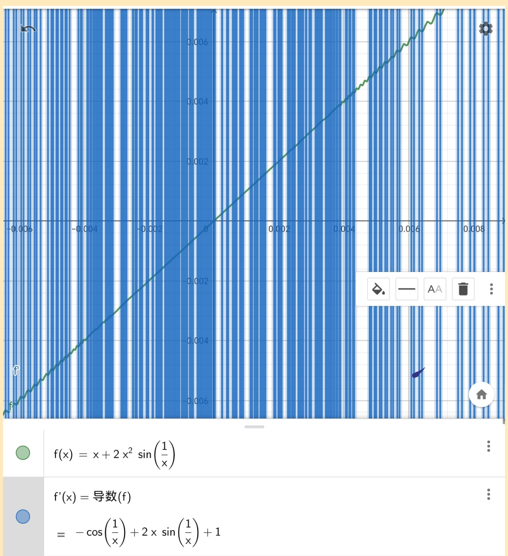
对于 ① 有 $f(x)=\begin{cases} x^2\sin^2\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

② 有 $f(x)=\begin{cases} x+2x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

① $f'(0)=\dfrac{x^2\sin^2\frac{1}{x}-0}{x}=x\sin^2\dfrac{1}{x}=0$

$f'(x)=2x\sin^2\frac{1}{x}+2\sin\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)\cdot x^2\cdot\cos\frac{1}{x}=2x\sin^2\frac{1}{x}-\sin\frac{2}{x}$，$x\neq 0$

0 处振荡



② $f'(0)=1$

原函数无法用初等函数表示的一些例子.

$$\int e^{-x^2}dx$$

$$\int \frac{\sin x}{x}dx$$

$$\int \frac{\cos x}{x}dx$$

$$\int \frac{1}{\ln x}dx$$

$$\int \frac{e^x}{x}dx$$

$$\int \ln(\sin x)dx$$

$$\int \sin x^2 dx$$

求 $\int \frac{1}{1+x^6}dx$

$$= \int \frac{1+x^2-x^2}{(1+x^2)(x^4-x^2+1)}dx$$

$$= \int \frac{1}{x^4-x^2+1} - \frac{1}{3}\int \frac{dx^3}{1+x^6}$$

Trip

$$= \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}$$

$$\left(x^2+\frac{1}{x^2}\right)-1 = \left(x+\frac{1}{x}\right)^2-3 = \left(x-\frac{1}{x}\right)^2+1.$$

$$d\left(x+\frac{1}{x}\right) = 1-\frac{1}{x^2} \qquad d\left(x+\frac{1}{x}\right) = 1+\frac{1}{x^2}$$

故 $\int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1} = \frac{1}{2}\left(\int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+1} - \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-3}\right).$

Δ_1, Δ_2 是 $[a,b]$ 的两个分割 若 Δ_2 由 Δ_1 添加 k 个分点, 形成的分割, 则

$0 \le \bar{S}_{\Delta_1} - \bar{S}_{\Delta_2} \le k\|\Delta\|(M-m) \qquad 0 \le \underline{S}_{\Delta_2} - \underline{S}_{\Delta_1} \le k\|\Delta\|(M-m)$

/合并分割 $\Delta^* = \Delta_1 \cup \Delta_2$. 加细分割 $\Delta_2 \subset \Delta_1$ (细)

pf: 设 $k=1$

$$\bar{S}_{\Delta_1} - \bar{S}_{\Delta_2} = M_{k_0}(x_{k_0}-x_{k_0-1}) - \left(M_{k_0}^1(x_{k'}-x_{k_0-1}) + M_{k_0}^2(x_k - x_{k'})\right)$$

① $M_{k_0}^1$, $M_{k_0}^2 < M_{k_0}$ 得左式

② $M_{k_0}^i < M \qquad M_{k_0}^1, M_{k_0}^2 > m$ 得 右 k 到时 重复 k 个 $k=1$ 情形

定义 1.4 (Darboux)

设 $f(x)$ 是 $[a,b]$ 上的有界函数, 对 $[a,b]$ 的任一分割 Δ, 作相应的上

和数集$\{\overline{S}_\Delta\}$与下和数集$\{\underline{S}_\Delta\}$，且记其下、上确界各为

$$\inf_{\Delta}\{\overline{S}_\Delta\} = \overline{\int_a^b} f(x)dx, \quad \sup_{\Delta}\{\underline{S}_\Delta\} = \underline{\int_a^b} f(x)dx$$

并各称为$f(x)$ 在$[a,b]$上的上积分，下积分。

$\overline{\int_a^b} f(x)dx < \overline{S}_\Delta < \overline{\int_a^b} f(x)dx + \varepsilon$ 　　 $\underline{\int_a^b} f(x)dx - \varepsilon < \underline{S}_\Delta < \underline{\int_a^b} f(x)dx$

证：对于下和 由上确界定义.

① $\forall \varepsilon$, $\exists \Delta_1$. $\underline{A} - \varepsilon < \underline{S}_{\Delta_1} < \underline{A}$. 取 $\sigma = \frac{\varepsilon}{|M-m|k}$ $\forall \|\Delta\| < \sigma$.

取 $\Delta^* = \Delta_1 \cup \Delta$ $0 \le \underline{A} - \underline{S}_\Delta = \underline{A} - \underline{S}_{\Delta_1} + \underline{S}_{\Delta_1} - \underline{S}_{\Delta^*} + \underline{S}_{\Delta^*} - \underline{S}_\Delta$ 　 k为Δ_1的分点.

$\underline{A} - \underline{S}_{\Delta_1} < \varepsilon$，$\underline{S}_{\Delta_1} - \underline{S}_{\Delta^*} \le 0$，$\underline{S}_{\Delta^*} - \underline{S}_\Delta < k\|\Delta\||M-m| < k\sigma|M-m|$

$\sum k\sigma|M-m| < \varepsilon$.

上确界同理

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(1) $f \in R[a,b] \Leftrightarrow$ (2) $\overline{A} = \underline{A} = A \Leftrightarrow$ (3) $\forall \varepsilon > 0$, $\exists \sigma > 0$. $\|\Delta\| < \sigma$ ② $\overline{S}_\Delta - \underline{S}_\Delta < \varepsilon$. 任意分割

$\Updownarrow$ 　 $\Updownarrow$

f连续 or 只有有限间断点. $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \Delta$. ① s.t. $\overline{S}_\Delta - \underline{S}_\Delta < \varepsilon$ 找到一个

$= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$

$\Updownarrow$

$= \sum_{k=1}^{n} \omega_k(f)\Delta x_k$ → 振幅 ①—②的证明与附①类似.

单调 $\Leftrightarrow \forall \varepsilon, \forall \sigma, \Delta_a, w_i$之和对应的$< \sigma$

$\overline{S}_\Delta - \overline{S}_{\Delta^*} + \overline{S}_{\Delta^*} - \overline{S}_{\Delta_1} + \underline{S}_{\Delta_1} - \underline{S}_{\Delta^*} + \underline{S}_{\Delta^*} - \underline{S}_\Delta < 3\varepsilon$

$\overline{S}_\Delta - \overline{S}_{\Delta^*} < n(\sigma) = k\|\Delta\||M-m|$，$\overline{S}_{\Delta^*} - \overline{S}_{\Delta_1} \le 0$，$\underline{S}_{\Delta_1} - \underline{S}_{\Delta^*} < \varepsilon + \overline{S}_\Delta - \underline{S}_\Delta$，$\underline{S}_{\Delta^*} - \underline{S}_\Delta < n(\sigma)$，$k\|\Delta\||M-m|$

pf: f 连续 $\Rightarrow$ 一致连续. $\forall \varepsilon$, $\exists \delta$. 当 $|x_1 - x_2| < \delta$.

$|f(x_1) - f(x_2)| < \varepsilon$, 取 n 等分, $\frac{1}{n} < \delta$

则 $\sum_{i=1}^{n} \omega_i(f) \Delta x_i < \varepsilon(b-a) < \varepsilon'$

pf: f 单调, n 等分划分 $\sum_{i=1}^{n} \omega_i(f) \Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \frac{1}{n}$

不妨 $= \frac{b-a}{n}(f(a) - f(b))$, $n \to \infty$

---

pf: $f \in R[a,b]$. $\Rightarrow$ $\overline{A} = \underline{A}$. $\Rightarrow$

① 设 $\lim_{\|T\| \to 0} \sum_{k=1}^{n} f(\xi_k) \Delta x_k = A$

上确界 $\quad$ $\overline{S}_\delta$ 的下确界

$A + \varepsilon > \sum_{i=1}^{n} f(\xi_k') \Delta x_k > \overline{S}_\delta - \varepsilon > \overline{A} - \varepsilon$

下确界 $\quad$ $\underline{S}_\delta$ 的上确界

$A - \varepsilon < \sum_{i=1}^{n} f(\xi_k') \Delta x_k < \underline{S}_\delta + \varepsilon \leq \underline{A} + \varepsilon$

$A - \varepsilon \leq \underline{A} + \varepsilon \leq \overline{A} + \varepsilon \leq \overline{A} - \varepsilon + 2\varepsilon$

$< A + 3\varepsilon$

$|\overline{A} - \underline{A}| \leq 4\varepsilon$

② $\overline{A} = \underline{A} \Rightarrow f \in R[a,b]$

$A-\varepsilon < \underline{S}_\Delta < S < \overline{S}_\Delta < A+\varepsilon$ ✓

$$\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{\omega_i<\varepsilon} \omega_i \Delta x_i + \sum_{\omega_i\ge\varepsilon} \omega_i \boxed{\Delta x_i}$$ 总长为 S_ε

① $S_\varepsilon<\sigma$ 时 左边 $< \sum_{i=1}^{n} \varepsilon \Delta x_i + (M+m)\sigma$ · 任意小.

② $f\in R$ 时

$\varepsilon\sigma >$ 左边 $> \sum_{\omega_i>\varepsilon} \omega_i \Delta x_i > \varepsilon \sum \Delta x_i$ pf up

---

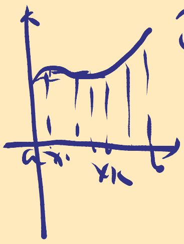
1法. 2法, 3法. 单调可积无限间断点.

若 $f(x)\in R[a,b]$

则 ① ∃一个阶梯函数 $h(x)$ s.t. $\int_a^b |f(x)-h(x)|dx < \varepsilon$

② ∃一个连续函数 $g(x)$ s.t. $\int_a^b |f(x)-g(x)|dx < \varepsilon$

pf:



① 取 $h(x)$ 在 $[x_{i-1}, x_i]$ 上为 $f(x)$ 的下确界.

则 $\int_a^b |f(x)-h(x)|dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (f(x)-h(x))dx < \sum_{i=1}^{n-1} \omega(f) dx$

又由 $f(x)\in R[a,b]$ 成立

② 取分点的端点值连线. $f(x)-g(x)\le \omega_i(f)$

---

pf: $\lim_{n\to\infty} \int_0^1 (1+x^n)^\alpha dx$

$$= \int_0^{1-\varepsilon} (1+x^n)^\alpha dx + \int_{1-\varepsilon}^{1} (1+x^n)^\alpha dx$$

$$\le \int_0^{1-\sigma}(1+(1-\sigma)^n)^{\alpha}dx + \int_{1-\sigma}^{1}2^{\alpha}dx$$

$$\le (1+(1-\sigma)^n)^{\alpha}\cdot 1 + 2^{\alpha}\cdot\sigma.$$ 又 $(1+x)^{\alpha}\sim \alpha x+1$

$$\sim 1+\alpha(1-\sigma)^n+2^{\alpha}\sigma$$ 取 $\sigma<\frac{\varepsilon}{2^{\alpha+1}}$

$$< 1+\frac{\varepsilon}{2}+\alpha\left(1-\frac{\varepsilon}{2^{\alpha+1}}\right)^n;$$ 当n足够大，$\alpha\left(1-\frac{\varepsilon}{2^{\alpha+1}}\right)<\frac{\varepsilon}{2}$

Wallis 公式

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$ $$\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)\frac{1}{2n+1}=\frac{\pi}{2}$$

n为偶 $\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{n-1}{n}=\frac{\pi}{2}\cdot\frac{(2m-1)!!}{2m!!}$

n为奇 $\frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdots\frac{n-1}{n}=\frac{(2m)!!}{(2m+1)!!}$

Pf

① $\underbrace{\int_0^{\frac{\pi}{2}}\sin^n x\,dx}_{D}$ 令 $x=\frac{\pi}{2}-t$ 则有 $x\in(0,\frac{\pi}{2})$ $t\in(\frac{\pi}{2},0)$

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_{\frac{\pi}{2}}^{0}\sin^n\left(\frac{\pi}{2}-t\right)d\left(\frac{\pi}{2}-t\right)=\int_0^{\frac{\pi}{2}}\cos^n t\,dt=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

② 对于 $\underbrace{\int_0^{\frac{\pi}{2}}\sin^n x\,dx}_{I_n}=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d\cos x$

$$=-\left(\sin^{n-1}x\cos x-\int_0^{\frac{\pi}{2}}(n-1)\cdot\sin^{n-2}x\,\cos x\cdot\cos x\,dx\right)$$

$$=\left(-\sin^{n-1}\cos x\right)\Big|_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

即 $I_n\cdot n=(n-1)\cdot I_{n-2}$

$I_0=\frac{\pi}{2}$ $\quad I_1=1$

$I_2=\frac{\pi}{2}\cdot\frac{1}{2}$ $\quad I_3=\frac{2}{3}$

$I_4=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}$ $\quad I_5=\frac{2}{3}\cdot\frac{4}{5}$

$I_{2m}=I_{2m-2}\cdot\frac{2m-1}{2m}$ $\quad I_{2m+1}=I_{2m-1}\cdot\frac{2m}{2m+1}$

③ 由于 $\sin^{2n+1}x<\sin^{2n}x>\sin^{2n+1}x$ Wallis 公式

$$\frac{(2n-2)!!}{(2n-1)!!}>\frac{\pi}{2}\cdot\frac{(2n-1)!!}{2n!!}>\frac{2n!!}{(2n+1)!!}\quad\frac{1}{2n}>\frac{\pi}{2}\cdot\frac{(2n-1)!!^2}{(2n!!)^2}>\frac{1}{2n+1}$$

故 $\lim\limits_{n\to\infty}\frac{\pi}{2}\left(\frac{(2n-1)!!}{(2n!!)}\right)^2(2n+1)=1$

Cauchy schwarz 不等式

$$\int_a^b|f(x)\cdot g(x)|\le\left(\int_a^b f^2(x)\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\right)^{\frac{1}{2}}$$

pf: 令 $h(x)=f(x)+tg(x)$

$$\int_a^b h(x)^2=\int_a^b f^2(x)dx+2t\int_a^b f(x)g(x)+t^2\int_a^b g^2(x)dx$$

故判别 $\Delta\le0$

得证

内积形式为 $|(\alpha,\beta)|\le|\alpha|\cdot|\beta|$

pf: 令 $\gamma=\alpha+t\beta$.

$(\gamma,\gamma)=(\alpha+t\beta,\alpha+t\beta)$

$=(\alpha,\alpha)+2t(\alpha,\beta)+t^2(\beta,\beta)$

≥0

关于 t 的二次式 ≥0 $\quad\Delta\le0$.

$4(\alpha,\beta)^2-4(\alpha,\alpha)(\beta,\beta)\le0$

故 $|(\alpha,\beta)|\le|\alpha||\beta|$

Minkowski不等式

$$\left(\int_a^b [f(x)+g(x)]^2 dx\right)^{\frac{1}{2}} \leq \left(\int_a^b f(x)^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b g^2(x) dx\right)^{\frac{1}{2}}$$

运用Cauchy Schwarz

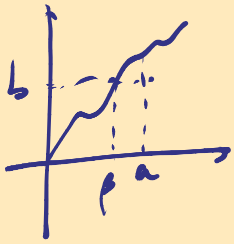
$$\int_a^b [f(x)+g(x)]^2 dx = \int_a^b f^2(x) dx + 2\int_a^b f(x)g(x) + \int_a^b g^2(x) dx$$

$$\leq \int_a^b f^2(x) dx + 2\left(\int_a^b f^2(x) dx\right)^{\frac{1}{2}}\left(\int_a^b g^2(x) dx\right)^{\frac{1}{2}} + \int_a^b g^2(x) dx$$

$$= \left[\left(\int_a^b f^2(x) dx\right)^{\frac{1}{2}} + \left(\int_a^b g^2(x) dx\right)^{\frac{1}{2}}\right]^2$$

Young不等式

$f(x) \in C[0,\infty)$ $f'(x)>0$, $f(0)=0$. 则 $\int_0^a f(x)dx + \int_0^b f^{-1}(y)dy \geq ab$ (几何意义明显)



证. 令 $F(x) = \int_0^x f(x)dx + \underbrace{\int_0^b f^{-1}(y)dy}_{\text{常数}} - xb$

$F'(x) = f(x) - b$

令 $f(\beta)=b$. ∵ $f(x)\nearrow$ 故 β 为 $F(x)$ 极小值点.

$F(x)_{min} = F(\beta) = \int_0^\beta f(x)dx + \int_0^b f^{-1}(y)dy - \beta b$. 将 $y=f(x)$ 代入则有

$$= \int_0^\beta f(x)dx + \int_0^\beta x f'(x)dx - \beta b$$

$$= x f(x)\Big|_0^\beta - \beta b$$

故 $F(a) \geq F(\beta) = 0$

$$= \beta b - \beta b = 0$$

Riemann-lebesque 引理

$f \in R[a,b]$ 则 $\lim\limits_{x\to\infty}\int_a^b f(x)\sin\lambda x\,dx = 0$

Rieman 函数 $R(x) = \begin{cases} 0 & x \in \mathbb{Q}^c \\ \frac{1}{q} & x \in \mathbb{Q},\ x=\frac{p}{q}.\ p,q\text{互质} \\ 1 & x=0 \end{cases}$

附1. $R(x)$在无理点连续,有理点为第三类间断点

即证 $\forall \varepsilon>0$, $\exists \delta$, $x\in U(x_0,\delta)$. $|f(x)-f(x_0)| = |f(x)| < \varepsilon$, x_0为无理点.

要 s.t $f(x) = \frac{1}{q} < \varepsilon$ 即 $q > [\frac{1}{\varepsilon}]+1 > \frac{1}{\varepsilon}$ ∵ $q \leq [\frac{1}{\varepsilon}]+1$ 的点只有有限个,设其为 $x_1 \cdots x_n$.

取 $\delta < \min\{|x_i - x_0|\}$ 则 $x\in(x_0,\delta)$ 中均有 $\frac{1}{q}<\varepsilon$. 故 $f(x)<\varepsilon$ 证毕

附2 $R(x) \in R[0,1]$

要证 $R(x)\in R[0,1]$. 可以考虑 $\forall \varepsilon$, $\omega_i(f) > \varepsilon$ 的区间和可以使其任意小

∵ $\frac{1}{q} > [\frac{1}{\varepsilon}]+1 > \frac{1}{\varepsilon}$ 的点只有有限个. 实际上由有理点和无理点的稠性 $\omega_i = \frac{1}{q}$

$$\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{i=1}^{k_1} \omega_i \Delta x_i + \sum_{i=1}^{k_2} \omega_i' \Delta x_i$$,其中 $\omega_i < \varepsilon$, $\omega_i' \geq \varepsilon$, 设其中最大的为 $\frac{1}{Q}$

$$\leq \varepsilon + \frac{1}{Q}\sum_{i=1}^{k_2}\Delta x_i$$ 其中 k_2是有限的 最多只有 k 个

$$\leq \varepsilon + \frac{1}{Q} k_2 \|\Delta\|$$,故取 $\|\Delta\| \leq \frac{Q\varepsilon}{k_2}$ 则有

$$\leq 2\varepsilon$$ 故 $R(x)\in R[0,1]$

Stirling公式 $n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \quad (n\to\infty)$

附:

设 $f(x)\in C^1[0,a]$, $f(0)=0$ 则附 $\int_0^a |f(x)\cdot f'(x)|\,dx \le \frac{a}{2}\int_0^a (f'(x))^2dx$

令 $F(x)=\int_0^x |f'(t)|\,dt$ 则 $F'(x)=|f'(x)|$　$F(x)\ge |f(x)|$

$f(x)=\int_0^x f'(t)\,dt \le F(x)$

故 $\int_0^a |f(x)f'(x)|\,dx \le \int_0^a F(x)\cdot F'(x)\,dx = \frac{F^2(x)}{2}\Big|_0^a = \frac{1}{2}F^2(a)$

而 $F(a)\le \int_0^a |f'(t)|\,dt \le \left(\int_0^a 1\,dx\right)^{\frac12}\cdot\left(\int_0^a (f'(t))^2dt\right)^{\frac12}$

$= \sqrt{a}\left(\int_0^a f'^2(x)\,dx\right)^{\frac12}$

故左式 $\le \frac{a}{2}\int_0^a (f'(x))^2dx$

变积分数的法

$f(x)\le \int_0^x |f'(x)|\,dx$

求 $\underbrace{\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx}_{A}$,

令 $x=\pi-t$

即有 $\int_\pi^0 \frac{(\pi-t)\sin t}{1+\cos^2 t}d(\pi-t)=\int_0^\pi\frac{(\pi-t)\sin t}{1+\cos^2 t}dt=\underbrace{\int_0^\pi\frac{(\pi-x)\sin x}{1+\cos^2 x}dx}_{B}$

故 $A=B$. $A=\frac{A+B}{2}=\int_0^\pi\frac{\pi\sin x}{1+\cos^2 x}dx=-\int_0^\pi\frac{\pi}{1+\cos^2 x}d\cos x$

$=-\pi\arctan\cos x\Big|_0^\pi$

Tip $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

↓

$$\underbrace{\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^2 x}{x(\pi-2x)}dx}_{A}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2 x}{(\frac{\pi}{2}-x)(\pi-2(\frac{\pi}{2}-x))}dx=\underbrace{\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2 x}{x(\pi-2x)}dx}_{B}$$

故 $A=\frac{A+B}{2}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{x(\pi-2x)}dx=\frac{2}{\pi}\int_0^a\left(\frac{1}{2x}+\frac{1}{\pi-2x}\right)dx$

设 $f(x)\in C[0,1]$ 非负, 且 $f^2(x)\le 1+2\int_0^x f(t)dt$ 证: $f(x)\le 1+x$ $(x\in[0,1])$

证: 令 $1+2\int_0^x f(t)dt=F(x)$. $f(x)\le\sqrt{F(x)}$. $F'(x)=2f(x)$

故 $\frac{F'(x)}{2}\le\sqrt{F(x)}$ 即 $\frac{F'(x)}{2\sqrt{F(x)}}\le 1$. 故 $\int_0^x\frac{F'(x)}{2\sqrt{F(x)}}dx=\sqrt{F(x)}\Big|_0^x\le x$

故 $\sqrt{F(x)}-\sqrt{F(0)}\le x$. $F(0)=1$ 故 $\sqrt{F(x)}\le 1+x$ 故

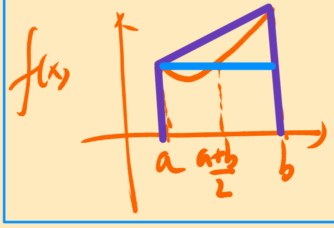
$f(x)\le 1+x$

Hardamar 不等式

→ 这部分 证为 Jasen不等式

$f(x)\in C[a,b]$，且 $f''(x)>0$ 则有 $f\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$

直观理解 即面积的大小关系



帅. 左 Ta

$$f(x)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)+\frac{f''(\xi)}{2}\left(x-\frac{a+b}{2}\right)^2$$

$f''(x)>0$. 则 $f(x)>f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$

两边对 $a\sim b$ 取积分.

$$\boxed{\int_a^b f(x)\,dx>(b-a)f\left(\frac{a+b}{2}\right)}+\boxed{\int_a^b f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)dx}$$

故左边成立.

令 $x-\frac{a+b}{2}=t$

$\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} f'\left(\frac{a+b}{2}\right)t\,dx$ 奇函数 $=0$

帅. 右 即证

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx>0$$

令 $F(x)=\frac{f(a)+f(x)}{2}(x-a)-\int_a^x f(t)\,dt$

$$F'(x)=\frac{f'(x)}{2}(x-a)+\frac{f(a)+f(x)}{2}-f(x)$$

$$F''(x)=\frac{f''(x)}{2}(x-a)+\frac{f'(x)}{2}+\frac{f'(x)}{2}-f'(x)$$

$$=\frac{f''(x)}{2}(x-a)\geqslant 0$$

故 $F'(x)\nearrow$ $F'(x)>F'(a)=0$

故 $F(x)\nearrow$ $F(x)>F(a)=0$. ✓

用第一中值定理
可不行哦

例 1.34 (带积分余项的Taylor 公式)

$$
\begin{aligned}
f(x) - f(x_0) &= \int_{x_0}^{x} f'(t)dt = \int_{x_0}^{x} f'(t)d(t-x) \\
&= f'(x_0)(x-x_0) - \int_{x_0}^{x} f''(t)(t-x)dt \\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{1}{2}\int_{x_0}^{x} f'''(t)(t-x)^2 dt \\
&= \cdots \\
&= \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt
\end{aligned}
$$

第一中值定理

$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$.

f(x)连续

g(x)在[a,b]不变号

① 整体运用中值定理

$= \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n(x-x_0)$ (Cauchy)

② 部分

$= \frac{1}{n!} f^{(n+1)}(\xi)\int_{x_0}^{x}(x-t)^n dt$

$= \frac{1}{n!} f^{(n+1)}(\xi)(x-x_0)^{n+1}$ (Lagrange)

$\lim_{n\to\infty}\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)}{x}dx = f(0)\ln\frac{b}{a}$ 用中值定理 $\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)}{x}dx = f(\xi)\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{1}{x}dx = f(\xi)\ln\frac{b}{a}$

$n\to\infty$ $\xi\in(\frac{a}{n},\frac{b}{n})\to 0$. ✓

例 1.36

设 $f(x) \in C^2[-1,1]$, $f(0)=0$, 则 $\exists\xi\in[-1,1]$, 使得

$$
f''(\xi) = 3\int_{-1}^{1} f(x)dx.
$$

令 $F(x)=\int_{-1}^{x} f(x)\,dx$ $F(x)\in C^3[-1,1]$

即证 $F^{(3)}(\xi)=3\big(F(1)-F(-1)\big)$ $F'(0)=0=f(0)$

$$F(x)=F(0)+\frac{F''(0)}{2!}x^2+\frac{F'''(\xi)}{3!}(x)^3$$

$$F(1)=F(0)+\frac{F''(0)}{2!}+\frac{F'''(\xi_1)}{3!}$$

$$F(-1)=F(0)+\frac{F''(0)}{2!}+\frac{F'''(\xi_{-1})}{3!}\cdot(-1)$$

$$3\big(F(1)-F(-1)\big)=\boxed{\frac{F'''(\xi_1)+F'''(\xi_{-1})}{2}}$$ 由连续函数性质

例 1.37

设$f(x) \in C[0,\pi]$，且$\int_0^\pi f(x)dx = 0$, $\int_0^\pi f(x)\cos x dx = 0$

则$\exists \xi_1, \xi_2 \in (0,\pi)$，使得$f(\xi_1) = f(\xi_2) = 0$.

$$\int_0^\pi f(x)\cos x\,dx = \underbrace{\cos x F(x)\Big|_0^\pi}_{0} + \underbrace{\int_0^\pi F(x)\sin x\,dx}_{第一中值}, \quad F(x) = \int_0^x f(x)\,dx$$

$$\underline{F(\pi) = \int_0^\pi f(x)dx = 0}$$

$$= F(\xi)\int_0^\pi \sin x\,dx = 0$$

$$F(0) = F(\xi) = F(\pi) = 0$$

$$f(\xi_1) = f(\xi_2) = 0 \quad \text{Roll中值定理}$$

定积分第二中值定理

定理 1.24 (Bonnet型)

设$g(x) \in R[a,b]$。 (按最大的缩小一点..)

(1) 若$f(x)$ 是$[a,b]$ 上的非负递减函数，则存在$\xi \in [a,b]$，有$\displaystyle\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx.$

(2) 若$f(x)$ 是$[a,b]$ 上的非负递增函数，则存在$\xi \in [a,b]$，有$\displaystyle\int_a^b f(x)g(x)dx = f(b)\int_{\xi}^{b} g(x)dx.$

定理 1.25 (Weierstrass型)

设$f(x)$在$[a,b]$上单调，$g(x) \in R[a,b]$。则存在$\xi \in [a,b]$，使得

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx + f(b)\int_{\xi}^{b} g(x)dx.$$

定积分第二中值定理

引理 1.26 (Abel 变换) (离散版本的分部积分)

设有两组数$\{a_k\}_{k=1}^n, \{b_k\}_{k=1}^n$，记$A_k = \sum_{i=1}^{k} a_i, (1 \le k \le n)$，则

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} A_k(b_k - b_{k+1}) + A_n b_n. \quad 令 A_0 = 0$$

$$= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} A_{k-1} b_k. \quad j = k-1$$

再代换

推论 1.27

若有$m \le A_k \le M, (1 \le k \le n)$，且$b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$，则有

$$m b_1 \le \sum_{k=1}^{n} a_k b_k \le M b_1.$$

定积分第二中值定理应用

例 1.38

设$f(x) \in D[a,b]$，且$f'(x)$ 是单调递减函数，$f'(b) \geq m > 0$. 则

$$\left|\int_a^b \cos f(x)dx\right| \leq \frac{2}{m}.$$

$f'(x) \nearrow$

$= \int_a^b \frac{f'(x)}{f'(x)} \cos f(x)\, dx = \frac{1}{f'(b)} \int_\xi^b f'(x)\cos f(x)\, dx$

$\le \frac{1}{m} \left|\sin f(x)\Big|_\xi^b\right| \le 2$

$\le \frac{2}{m}$

例 1.39 (Riemann-Lebesque 引理)

设$f(x)$ 是$[a,b]$ 上的单调函数, 则

$$\lim_{\lambda\to+\infty}\int_a^b f(x)\sin\lambda x dx = 0.$$

$$\lim_{\lambda\to+\infty}\int_a^b f(x)\cos\lambda x dx = 0.$$

$= f(a)\int_a^\xi \cos\lambda x\, dx + f(b)\int_\xi^b \cos\lambda x\, dx$

$$= f(a)\frac{\sin\lambda x}{\lambda}\Big|_a^{\xi} + f(b)\frac{\sin\lambda x}{\lambda}\Big|_{\xi}^{b} \quad \lambda\to+\infty, \to 0.$$

若$f(x)$可导，则证明可直接使用分部积分完成

$f(x) \in [a,b]$ ↗ 则 $\int_a^b x f(x)\,dx \geq \frac{a+b}{2}\int_a^b f(x)\,dx.$

证1. $f(x)\nearrow$

$$\left(x-\frac{a+b}{2}\right)\left(f(x)-f\left(\frac{a+b}{2}\right)\right)\geq 0$$

积分，

$$\int_a^b \left(x-\frac{a+b}{2}\right)(f(x))\,dx - \underbrace{\int_a^b \left(x-\frac{a+b}{2}\right)f\left(\frac{a+b}{2}\right)}_{=0} \geq 0$$

证2. 使用第二中值定理

$$f(a)\leq f(b)$$

只要证

$$\int_a^b \left(x-\frac{a+b}{2}\right)f(x)\,dx$$

$$= f(a)\int_a^{\xi}\left(x-\frac{a+b}{2}\right)dx + f(b)\int_{\xi}^{b}\left(x-\frac{a+b}{2}\right)dx$$

$$= f(a)\left(\frac{x^2}{2}-\frac{a+b}{2}x\right)\Big|_a^{\xi} + f(b)\left(\frac{x^2}{2}-\frac{a+b}{2}x\right)\Big|_{\xi}^{b}$$

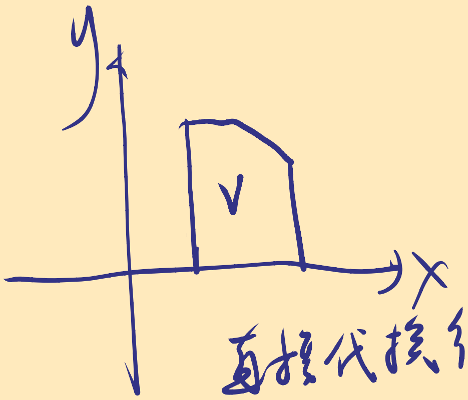
$$= f(a)\left(\left(\frac{\xi^2}{2}-\frac{a+b}{2}\xi\right)+\frac{ab}{2}\right) + f(b)\left(-\frac{ab}{2}-\left(\frac{\xi^2}{2}-\frac{a+b}{2}\xi\right)\right)$$

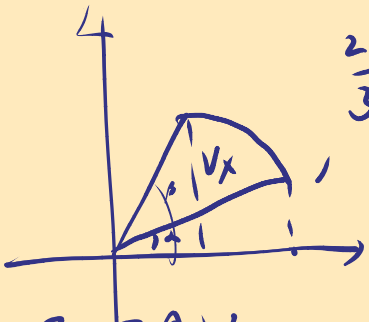
$\xi\in[a,b]$ $$= \frac{f(a)}{2}(\xi-a)(\xi-b) - \frac{f(b)}{2}(\xi-a)(\xi-b)$$

$$= \frac{1}{2}\cdots = (f(a)-f(b))(\xi\cdots$$

	直角坐标显式方程 $y=f(x), x\in[a,b]$	直角坐标参数方程 $\begin{cases} x=x(t), \\ y=y(t), \end{cases} t\in[T_1,T_2]$	极坐标方程 $r=r(\theta), \theta\in[\alpha,\beta]$
平面图形面积	$\int_a^b f(x)\,dx$	$\int_{T_1}^{T_2} \lvert y(t)x'(t)\rvert\,dt$	$\frac{1}{2}\int_\alpha^\beta r^2(\theta)\,d\theta$
弧长的微分	$dl=\sqrt{1+[f'(x)]^2}\,dx$	$dl=\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$	$dl=\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$
曲线弧长	$\int_a^b \sqrt{1+[f'(x)]^2}\,dx$	$\int_{T_1}^{T_2} \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$	$\int_\alpha^\beta \sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$
旋转体体积	$\pi\int_a^b [f(x)]^2\,dx$	$\pi\int_{T_1}^{T_2} y^2(t)\lvert x'(t)\rvert\,dt$	$\frac{2}{3}\pi\int_\alpha^\beta r^3(\theta)\sin\theta\,d\theta$
旋转曲面面积	$2\pi\int_a^b \lvert f(x)\rvert\sqrt{1+[f'(x)]^2}\,dx$	$2\pi\int_{T_1}^{T_2} \lvert y(t)\rvert\sqrt{x'^2(t)+y'^2(t)}\,dt$	$2\pi\int_\alpha^\beta r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

大部分都可以直接代换得到，但极坐标下的不行





$\frac{2}{3}\pi\int_\alpha^\beta r^3\sin\theta\,d\theta$

直接代换得到的是V而不是Vx

曲率 $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $\qquad \varphi=\arctan\frac{y'(t)}{x'(t)}$

$$K=\frac{d\varphi}{ds}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{x'(t)^2+y'^2(t)}$$

已知 $\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$，求 $\int_0^{+\infty}\frac{x-\sin x}{x^3}dx$.

若拆开 则 $\int_0^{+\infty}\frac{1}{x^2}dx-\int_0^{+\infty}\frac{\sin x}{x^3}dx$ $\quad\frac{1}{x^2}$的积分发散.

发散－收敛≠收敛. 故不能拆开做

则 $\int_0^{+\infty}\frac{x-\sin x}{x^3}dx=-\frac{1}{2}\int_0^{+\infty}(x-\sin x)d\frac{1}{x^2}$

$$=-\frac{1}{2}\left(\underbrace{\frac{x-\sin x}{x^2}\Bigg|_0^{+\infty}}_{0}-\int_0^{+\infty}\frac{1-\cos x}{x^2}dx\right)$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{1-\cos x}{x^2}dx$$

$$=\frac{1}{2}\left((1-\cos x)(-\frac{1}{x})\Bigg|_0^{+\infty}+\int_0^{+\infty}\frac{\sin x}{x}dx\right)$$

$= \frac{\pi}{x}$

求积分 $\int_0^{+\infty} e^{-ax}\sin bx\,dx = I$, $\int_0^{+\infty} e^{-ax}\cos bx\,dx = J$. $(a>0)$

可以用两次分部积分.

也可以用 $\int_0^{+\infty} e^{(-a+ib)x}dx = \frac{1}{-a+ib}\left[e^{-ax}(\sin bx\cdot i+\cos bx)\right]_0^{+\infty}$

$e^{ix} = \cos x + i\sin x$

(可用Taylor公式时)

$$= \frac{1}{a-ib}$$

$$= \frac{a+ib}{a^2+b^2}$$

$$= J + iI$$

求积分 $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$ (Euler积分)

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

应用 $\int_0^{\frac{\pi}{2}} x\cot x\,dx = \int x\frac{\cos x}{\sin x}\,dx$

$$= \int x\,d\ln\sin x$$

$$= x\ln\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\ln\sin x\,dx$$

$$= \frac{\pi}{2}\ln 2$$

$I =$

故 $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$

$$= \int_0^{\frac{\pi}{2}} \ln\cos x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\cos x\,dx$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln\sin 2x\,dx - \frac{\pi}{2}\ln 2\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2x\,d2x - \frac{\pi}{2}\ln 2\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\int_0^{\pi} \ln\sin u\,du - \frac{\pi}{2}\ln 2\right)$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln\sin u\,du - \frac{\pi}{2}\ln 2\right)$$

故 $2I = I - \frac{\pi}{2}\ln 2$, $I = -\frac{\pi}{2}\ln 2$

求 $\int_0^{+\infty}\frac{\sin^4 x}{x^2}dx$

$$=-\frac{1}{x}\cdot\sin^4 x\Big|_0^{+\infty}+4\int_0^{+\infty}\frac{\sin^3 x\cos x}{x}$$

$$=\int_0^{+\infty}\frac{2\sin^2 x\cdot 2\sin x\cos x}{x}dx$$

$$=\int_0^{+\infty}\frac{(1-\cos 2x)\sin 2x}{x}dx$$

$$=\int_0^{+\infty}\frac{\sin 2x}{x}dx-\frac{1}{2}\int_0^{+\infty}\frac{\sin 4x}{x}dx$$

$$=\frac{\pi}{2}-\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

求 $\int_0^{+\infty}\frac{dx}{(1+x^2)(1+x^\alpha)}$ 分子分母次数相差大，想法是换元，将分子分母次数尽量靠近

令 $u=\frac{1}{x}$ 原式 $=I=\int_{+\infty}^{0}\frac{-\frac{1}{u^2}du}{(1+\frac{1}{u^2})(1+\frac{1}{u^\alpha})}=\int_0^{+\infty}\frac{u^\alpha du}{(1+u^2)(1+u^\alpha)}$

故 $2I=\int_0^{+\infty}\frac{dx}{1+x^2}=\arctan x\Big|_0^{+\infty}=\frac{\pi}{2}$ $I=\frac{\pi}{4}$

令 $x=\tan u$ $\int_0^{\frac{\pi}{2}}\frac{\sec^2 u\,du}{\sec^2 u(1+\tan^\alpha u)}=\int_0^{\frac{\pi}{2}}\frac{\cos^\alpha u\,du}{\sin^\alpha u+\cos^\alpha u}$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^\alpha u\,du}{\sin^\alpha u+\cos^\alpha u}$$

$2I=\frac{\pi}{2}$ $I=\frac{\pi}{4}$

反常积分收敛准则

① Cauchy收敛准则

$\forall \varepsilon>0,\ \exists G,\ \forall x_1, x_2>G,\ \left|\int_{x_1}^{x_2} f(x)\,dx\right|<\varepsilon$

② Dirichlet收敛准则

$\int_a^b f(x)\,dx$ 在 $[a,+\infty)$ 有界（亦即任意有限区间积分有界）

$g(x)$ 在 $[a,+\infty)$ 单调且 $\lim\limits_{x\to+\infty} g(x)\to 0$. 则 $\int_a^{+\infty} f(x)\cdot g(x)\,dx$ 收敛

③ Abel收敛准则

$\int_a^{+\infty} f(x)\,dx$ 收敛. $g(x)$ 在 $[a,+\infty)$ 单调有界 则 $\int_a^{+\infty} f(x)g(x)\,dx$ 收敛

Eg $\int_1^{+\infty} \frac{\sin x}{x^p}\,dx$ $\quad \int_0^{+\infty} \frac{\sin x}{x}\,dx = \underbrace{\int_0^1}_{R\text{积分}} + \int_1^{+\infty}$ 条件

① $p>1$ 时 $\quad \left|\frac{\sin x}{x^p}\right| < \frac{1}{x^p}$. $\int_1^{+\infty} \frac{1}{x^p}\,dx$ 在 $p>1$ 时绝对收敛.
$p\le 1$ 时发散.

故绝对收敛

② $p\in[0,1]$ 时 条件收敛

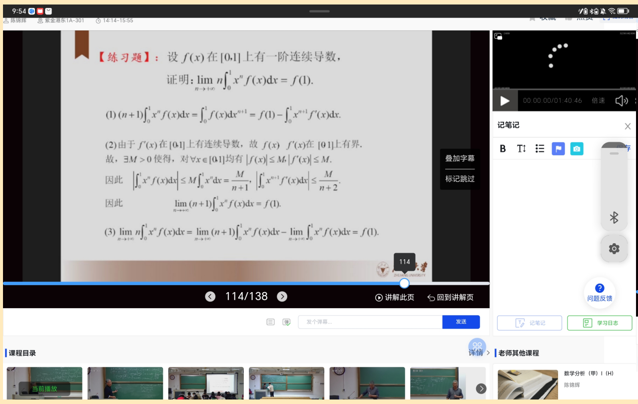
$\int_1^{+\infty} \frac{\sin x}{x^p}\,dx \quad \int_a^{+\infty} \sin x\,dx$ 任意有限区间有界

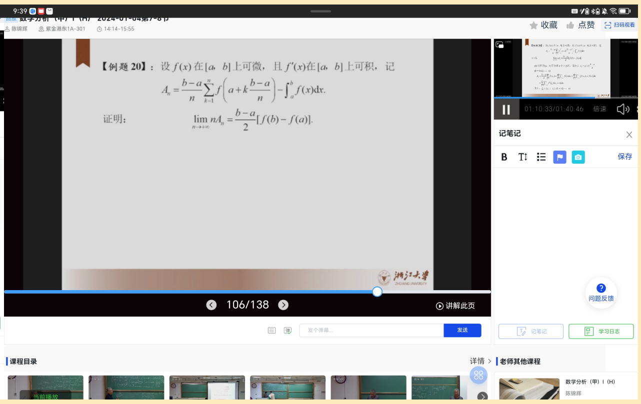
$\frac{1}{x^p} \downarrow \to 0$ 故收敛

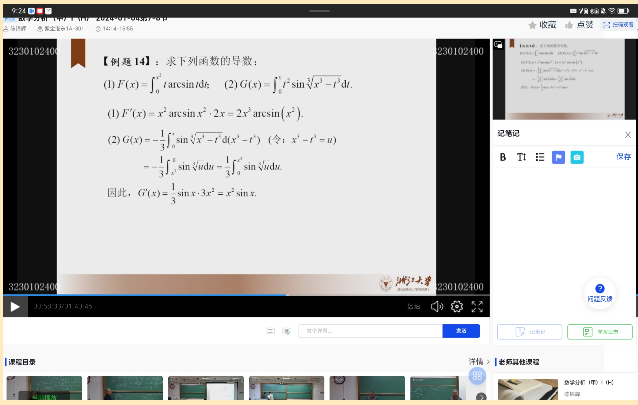
$\left|\frac{\sin x}{x^p}\right| > \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p} = \underbrace{\frac{1}{2x^p}}_{\text{发散}} - \underbrace{\frac{\cos 2x}{2x^p}}_{\text{收敛}}$

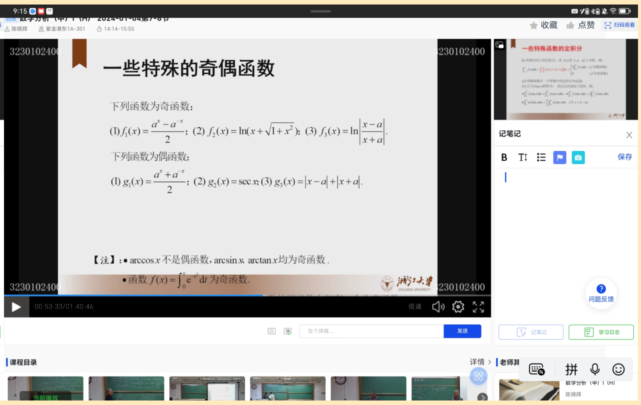
$$\int_0^{+\infty}\frac{\cos 2x}{2x^p}dx = 2^{p-1}\int_0^{+\infty}\frac{\cos 2x}{(2x)^p}d2x = A\int_0^{+\infty}\frac{\cos t}{t^p}dt$$

$\int_0^b \cos t\,dt$ 有界 $\frac{1}{t^p}\downarrow\to 0$ 故 $\int_0^{+\infty}\frac{\cos 2x}{2x^p}dx$ 收敛.









数分 II

$f(x) \in D^2 \cdot R$. 常数 C. s.t. $\sup(x^2|f(x)| \cdot |f''(x)|) \leq C$.

pf: $\exists M$. s.t $\sup(x|f'(x)|) \leq M$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2}h^2 \Rightarrow$$

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{f''(\xi)}{2}h \Rightarrow$$

$$\frac{1}{h}f'(x) = \frac{f(x+h)-f(x)}{h^2} - \frac{f''(\xi)}{2}. \quad 令 \ h = \frac{1}{x}.$$

$$|xf'(x)| = \sim \leq \left|x^2 f\left(x+\frac{1}{x}\right)\right| + |x^2 f(x)| + \left|\frac{f''(\xi)}{2}\right|$$

$$\left|x^2 f\left(x+\frac{1}{x}\right)\right| \leq \left|\left(x+\frac{1}{x}\right)^2 f\left(x+\frac{1}{x}\right)\right| = \frac{5}{2}C = M.$$

令 $t = -u$，求 min $f(x) = \int_{-1}^{1} |x-t| e^{t^2} dt$

$$f(x) = \int_{-1}^{1} |x+u| e^{u^2} du$$

$$f(x) = \frac{1}{2}\left(\int_{-1}^{1} |x+t| e^{t^2} dt + \int_{-1}^{1} |x-t| e^{t^2} dt\right)$$

$$\geq \int_{-1}^{1} |t| e^{|t|^2} dt$$

$$= 2\int_{0}^{1} t e^{t^2} dt$$

$$= \int_{0}^{1} e^{t^2} dt^2$$

$$= e^x \Big|_0^1$$

$$= e - 1$$