依概率收敛与依分布收敛的性质

(1) 若$\xi_n \xrightarrow{P} \xi$, $\eta_n \xrightarrow{P} \eta$, 则 $\xi_n \pm \eta_n \xrightarrow{P} \xi \pm \eta$;

(2) 若$\xi_n \xrightarrow{P} \xi$, $\eta_n \xrightarrow{P} \eta$, 则 $\xi_n\eta_n \xrightarrow{P} \xi\eta$;

(3) 若$\xi_n \xrightarrow{P} \xi$, $\eta_n \xrightarrow{P} c$, c 为常数, η_n 与c 都不为0, 则 $\xi_n/\eta_n \xrightarrow{P} \xi/c$;

(4) 设$\xi_n \xrightarrow{d} \xi$, $\eta_n \xrightarrow{P} c$, c 为常数, 则 $\xi_n + \eta_n \xrightarrow{d} \xi + c$, $\xi_n/\eta_n \xrightarrow{d} \xi/c$ $(c \neq 0)$.

(1) $P(|\xi_n \pm \eta_n - \xi \pm \eta| > \varepsilon) = P(|\xi_n - \xi \pm \eta_n - \eta| > \varepsilon)$ 小的发生,大的也一定发生

$\le P(|\xi_n - \xi| + |\eta_n - \eta| > \varepsilon)$

$\{|\xi_n - \xi| + |\eta_n - \eta| > \varepsilon\} \subseteq \{|\xi_n - \xi| > \frac{\varepsilon}{2}\} \cup \{|\eta_n - \eta| > \frac{\varepsilon}{2}\}$

$\le P(|\xi_n - \xi| > \frac{\varepsilon}{2}) + P(|\eta_n - \eta| > \frac{\varepsilon}{2})$

$\longrightarrow 0$

(2) $P(|\xi_n\eta_n - \xi\eta| > \varepsilon)$

$= P(|\xi_n\eta_n - \xi\eta_n + \eta_n\xi - \xi\eta| > \varepsilon)$

$= P(|\eta_n(\xi_n - \xi) + \xi(\eta_n - \eta)| > \varepsilon)$

$\le P(|\eta_n(\xi_n - \xi)| > \frac{\varepsilon}{2}) + P(|\xi(\eta_n - \eta)| > \frac{\varepsilon}{2})$

$\le \underbrace{P(|\eta_n| > M) + P(|\xi_n - \xi| > \frac{\varepsilon}{2M})} + \underbrace{P(|\xi| > M) + P(|\eta_n - \eta| > \frac{\varepsilon}{2M})}$.

至少发生一个一个.

任意 ε' 都∃ M, s.t. $P(|\eta_n| > M) < \frac{\varepsilon'}{4}$, ---- $P(\cdot) < \frac{\varepsilon'}{4}$

原式 $< \frac{\varepsilon'}{4} + \frac{\varepsilon'}{4} + \frac{\varepsilon'}{4} + \frac{\varepsilon'}{4} = \varepsilon'$

(3) $\xi_n/\eta_n \xrightarrow{P} \xi/C$.

只要证 $1/\eta_n \xrightarrow{P} \frac{1}{C}$

$$P\left(\left|\frac{1}{\eta_n} - \frac{1}{C}\right| > \varepsilon\right) = P\left(\frac{|\eta_n - c|}{|c\eta_n|} > \varepsilon\right)$$

$$\le P\left(|\eta_n| > \frac{|c|}{2}, |\eta_n - c| > \varepsilon C^2\right) + P\left(|\eta_n| \le \frac{|c|}{2}\right)$$

$$\le P(|\eta_n - c| > \varepsilon C^2) + P\left(|\eta_n - c| \ge \frac{|c|}{2}\right)$$

$$\longrightarrow 0$$

(4)

$\xi_n \xrightarrow{d} \xi$, $\eta_n \xrightarrow{P} C$

$\xi_n + \eta_n \longrightarrow \xi + C$ $\quad E e^{it(\xi_n+\eta_n)} \overset{\text{独立时}}{=} E e^{it\xi_n} \cdot E e^{it\eta_n}$

$$= E e^{it(\xi + C)}.$$

(Levy 定理).

若不独立，设 $F(x)$ 为 ξ 分布函数.

$$P(\xi_n + \eta_n \le x) - P(\xi + C \le x)$$

$$\le P(\xi_n + \eta_n \le x, \underline{|\eta_n - c| \le \varepsilon}) + P(|\eta_n - c| > \varepsilon)$$

$-F(x-c).$

$c-\varepsilon\le \eta_n \le c+\varepsilon$，则 $\xi_n+\eta_n > \xi_n + c-\varepsilon$

$$P(\xi_n+\eta_n\le x,\ |\eta_n-c|\le\varepsilon)\le P(\xi_n\le x-c+\varepsilon)$$

$$\le F_n(x-c+\varepsilon)-F(x-c)+P(|\eta_n-c|>\varepsilon).$$

$$\longrightarrow 0$$

同时

$$P(\xi_n+\eta_n\le x)-P(\xi+c\le x)$$

$$P(\xi+c\le x)\le P(\xi+c\le x,\ |\eta_n-c|<\varepsilon)+P(|\eta_n-c|\ge\varepsilon).$$

$$P(\xi_n+\eta_n\le x)\ge P(\xi_n+\eta_n\le x,\ |\eta_n-c|<\varepsilon).$$

$$\ge P(\xi_n+c+\varepsilon\le x).$$

故另一边

$$\ge F_n(x-c-\varepsilon)-F(x-c)-P(|\eta_n-c|\ge\varepsilon)$$

$$\to 0.$$ 夹逼

$$\xi_n/\eta_n \xrightarrow{d} \xi/c$$

$$P(\xi_n/\eta_n \le x) - P(\xi/c \le x)$$

$$\le P(\xi_n \le cx + c\varepsilon) + P(|\eta_n - c| > \varepsilon) - P(\xi \le cx)$$

$$= F_n(cx + c\varepsilon) - F(cx) + P(|\eta_n - c| > \varepsilon) \to 0$$

另一边同理

$$\ge F_n(cx - c\varepsilon) - F(cx) - P(|\eta_n - c| \ge \varepsilon)$$

$$\to 0$$